On the Interpretation of the Intergenerational Elasticity and the Rank-Rank Coefficients for Cross Country Comparison - Online Appendix

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In this section we follow Heckman et al. (2006) to construct the proof of Yitzhaki's theorem (Yitzhaki (1996)).

Theorem 1. (Yitzhaki's theorem) Let (Y, X) be i.i.d, assume $E[|X|], E[|Y|] < \infty$ assume that $E[Y|x]$ exists and is differentiable for every $x \in supp(X)$. Denote $\mu = E[X]$ and let $f(x)$ be the probability density function of X, then

$$\frac{\text{Cov}(Y, X)}{\text{Var}(X)} = \int_{-\infty}^{\infty} \frac{\partial E[Y|x = t]}{\partial t} w(t)dt,$$

where

$$w(t) = \frac{1}{\text{Var}(X)} \int_{t}^{\infty} (x - \mu)f(x)dx = \frac{1}{\text{Var}(X)} E[X - \mu|X > t]\text{P}(X > t)dx,$$

and the weights satisfy $\int_{-\infty}^{\infty} w(t) = 1$, $\lim_{t\to\infty} w(t) = 0$, $\lim_{t\to-\infty} w(t) = 0$,$\mu = arg\max_t w(t)$ and are increasing for $t < \mu$ and decreasing for $t > \mu$.

Proof. We follow Heckman et al. (2006)

$$\begin{aligned}\text{Cov}(Y, X) &= \text{Cov}(E[Y|x], X) \\ &= \int_{-\infty}^{\infty} E[Y|x = t](t - \mu)f_x(t)dt.\end{aligned}$$

Using integration by parts we have the following:

$$\int_{-\infty}^{\infty} E[Y|x=t](t-\mu)f_x(t)dt = \left[E[Y|x=t]\int_{-\infty}^{t}(u-\mu)f_x(u)du\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial E[Y|x=t]}{\partial t}\int_{-\infty}^{t}(u-\mu)f_x(u)dudt$$
$$= -\int_{-\infty}^{\infty}\frac{\partial E[Y|x=t]}{\partial t}E[X-\mu|X<t]\mathrm{P}(X<t)dt,$$

using the fact that

$$E[X-\mu] = 0 = E[X-\mu|X<t]\mathrm{P}(X<t) + E[X-\mu|X<t]\mathrm{P}(X>t)\mathrm{P}(X>t)$$

we obtain the following:

$$\mathrm{Cov}(Y,X) = \int_{-\infty}^{\infty}\frac{\partial E[Y|x=t]}{\partial t}E[X-\mu|X>t]\mathrm{P}(X>t)dt.$$

Therefore the weights are obtained as follows:

$$w(t) = \frac{1}{\mathrm{Var}(X)}\int_{t}^{\infty}(u-\mu)f(u)du = \frac{1}{\mathrm{Var}(X)}E[X-\mu|X>t]\mathrm{P}(X>t).$$

To see that the weights integrate to one, can employ integration by parts once more.

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty}(t-\mu)(t-\mu)f(t)dt = \left[(t-\mu)\int_{-\infty}^{t}(u-\mu)f(u)du\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\int_{-\infty}^{t}(u-\mu)f(u)dudt$$
$$= \int_{-\infty}^{\infty}\int_{t}^{\infty}(u-\mu)f(u)dudt$$

which implies $\int_{-\infty}^{\infty} w(t) = 1$. The definition of the weights reveal that the weights go to zero at the boundary of the support. To see that the weights are maximized at $t=\mu$, notice that for any $t<\mu$ we have the following:

$$\int_{t}^{\infty}(x-\mu)f(x)dx - \int_{\mu}^{\infty}(x-\mu)f(x)dx = \int_{t}^{\mu}(x-\mu)f(x)dx < 0$$

Similarly for any $t>\mu$ we have the following:

$$\int_{t}^{\infty}(x-\mu)f(x)dx - \int_{\mu}^{\infty}(x-\mu)f(x)dx = -\int_{\mu}^{t}(x-\mu)f(x)dx < 0$$

Finally, to see that the weights are increasing to the left of the mean and decreasing to its right, the first derivative is obtained as follows:

$$\frac{\partial w(t)}{\partial t} = -(t-\mu)f(t)$$

which is decreasing for every $t > \mu$ and increasing for every $t < \mu$. □

References

James J. Heckman, Sergio Urzua, and Edward Vytlacil. Understanding instrumental variables in models with essential heterogeneity. *The Review of Economics and Statistics*, 88(3):389–432, 2006.

Shlomo Yitzhaki. On using linear regressions in welfare economics. *Journal of Business & Economic Statistics*, 14(4):478–486, 1996.